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H^m convergence rates of solutions of GBBM equations in multi-dimensional spaces

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Abstract

In this paper, we study the decay rates of the generalized Benjamin–Bona–Mahony equations in n-dimensional space. By using Fourier analysis for long wave and by applying the energy method for short wave, we obtain the H^m convergence rates of the solutions when the initial data are in the bounded subset of the phase space $H^m(R^n)(n \geqslant 3)$. The optimal decay rates are obtained in our results and are found to be the same as the Heat equation.

Keywords: Generalized Benjamin–Bona–Mahony equations; Fourier analysis; Energy method; Optimal convergence rates

1. Introduction

The Benjamin–Bona–Mahony (BBM) equation and its counterpart, the Kortewegde Vries (KdV) equation, were both proposed as model equations for long waves in nonlinear dispersive media. The BBM equation was advocated and studied in [3] by Benjamin, Bona and Mahony. In [8], Guo proposed and studied a generalized BBM equation in multi-dimensions. The existence and uniqueness of solutions for GBBM equations have been proved by many authors (see [2,3,7,8]). The large time behaviors and decay rates of solutions to the initial value problems and the initial boundary value problems were also studied in [1,4,5,9]. But these studies do not give a clear picture of the decay rate in multi-dimensions. The goal of this paper is to give the optimal decay rates of solutions to Cauchy problems of the generalized BBM equations in multi-dimensional space.

We consider the system defined on $R^n(n \geqslant 3)$

$$u_t - \Delta u_t - \eta\Delta u + (\beta \cdot \nabla)u + div(f(u)) = 0, \tag{1}$$

where η is a positive constant and β is a real constant vector, $f(u) = (f_1(u), \dots, f_n(u))^\tau$.

The initial data are given by

$$u|_{t=0} = u_0(x). \tag{2}$$

In fact, we have given an optimal decay rate in the phase space $H^1(R^n)$ for the case of the nonlinear functions $f_j(u) = u^2$ in [6]. The paper is a follow-up of [6]. We obtained the optimal decay rates in the phase space $H^m(R^n)$ for the generalized nonlinear functions.

As we know, it is difficult to get the optimal decay rates of solutions using only energy estimates. Some new techniques are needed. As in [6], we give the frequency of decomposition for solutions. Then, by using the Fourier analysis for long wave and by applying the energy method for short wave to solutions, the L_2 convergence rates of the solution were obtained when the initial data are in the bounded subset of the phase space $H^m(R^n)$. This is the main difference between our study and related results of previous

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works. But how to obtain the H^m convergence rates of the solution is a new difficulty. We need more technology and more mature methods than those mentioned in [6].

Throughout this paper, we denote the generic constants as C. $W^{m,p}(R^n)$, $m \in Z_+$, $p \in [1,\infty]$, and we define the usual Sobolev space with the norm

$$\|f\|_{W^{m,p}} := \sum_{|\alpha|=0}^{m} \|\partial^\alpha f\|_{L_p}.$$

In particular, $W^{m,2} = H^m$.

As usual, Fourier transformation to the variable $x \in R^n$ is

$$\widehat{f}(\xi,t) \equiv (\mathcal{F}f)(\xi,t) = \int_{R^n} f(x,t)e^{-\sqrt{-1}x\xi}dx,$$

and the inverse Fourier transform to the variable ξ is

$$f(x,t) \equiv (\mathcal{F}^{-1}\widehat{f})(x,t) = (2\pi)^{-n}\int_{R^n} \widehat{f}(\xi,t)e^{\sqrt{-1}x\xi}d\xi.$$

By using the above notations, we can now state the main result of this paper as follows.

Theorem 1.1. *Given a ball* $\mathrm{B} = \{v \in H^{m+1}(R^n) \cap L^1(R^n) : \max(\|v\|_{L^1}, \|v\|_{H^{m+1}}) \leqslant R\}$ $(n \geqslant 3,\ m \geqslant [\frac{n}{2}] + 1,\ if\ k \geqslant 3;\ \geqslant 0\ if\ k = 2)$. *If the initial data are* $u_0(x) \in B$ *and* $f_j(u) = u^k$ $(j = 1,\cdots,n, k \in Z^+)$, *then there exists a solution* $u(x,t) \in L^\infty([0,\infty], H^{m+1})$ *to* (1), *defined globally in time. Moreover, for* $|\alpha| \leqslant m+1$

$$\|\partial_x^\alpha u(\cdot,t)\|_{L^2} \leqslant C(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}. \tag{3}$$

Remark 1. If $k = 2$, this theorem is a generalization of the result in [6].

Remark 2. The decay rate of solutions to Heat equations is also $(1+t)^{-\frac{n}{4}-\frac{|\alpha|}{2}}$.

The rest of the paper is arranged as follows. In Section 2, we will give the estimates for long wave of solutions by using Fourier analysis. In Section 3, the estimates for short wave of solutions obtained by using the energy method are given. Then, the time-asymptotic behavior of the solutions to (1) and (2) follows these estimates.

2. The estimates for long wave

In this section, we will study the estimates for long wave of solutions, using Fourier analysis. Let

$$\chi(\xi) = \begin{cases} 1, & |\xi| < \varepsilon, \\ 0, & |\xi| > 2\varepsilon, \end{cases} \tag{4}$$

be a smooth cut-off function, with ε being sufficiently small. Set $\chi(D)$ as a pseudodifferential operator with symbol $\chi(\xi)$. Define

$$u_L(x,t) = \chi(D)u(x,t), u_S(x,t) = (1-\chi(D))u(x,t).$$

Here, $u_L(x, t)$ and $u_S(x, t)$ represent long wave of solutions and short wave of solutions, respectively.

We will first recall some basic results for Fourier analysis.

Lemma 2.1 (Bernstein inequality). *For all* $1 \leqslant p \leqslant q < \infty$, *we have*

$$\|\chi(D)D_x^\alpha f\|_{L^q(R^n)} \leqslant C_{p,q}\varepsilon^{|\alpha|+n(1/p-1/q)}\|f\|_{L^p(R^n)},$$

where $C_{p,q}$ *is a constant depending only on p and q.*

Now we come back to considering (1). As in Ref. [6], applying the operator $\chi(D)$ to (1), we have

$$(u_L)_t - \Delta(u_L)_t - \eta\Delta u_L + (\beta\cdot\nabla)u_L + div(\chi(D)f(u)) = 0. \tag{5}$$

Define $\Phi(u) = -div\ \varphi(u)$ with $\varphi(u) = -(\chi(D)f(u))$; we can then write (5) as

$$(u_L)_t - \Delta(u_L)_t - \eta\Delta u_L + (\beta\cdot\nabla)u_L = \Phi(u).$$

For the long wave of the solutions, we first consider the Cauchy problem of the linear part of (5) as follows:

$$\begin{cases} E_t - \Delta E_t - \eta\Delta E + (\beta\cdot\nabla)E = 0, \\ E|_{t=0} = \delta(x), \end{cases} \tag{6}$$

where $\delta(x)$ is the Dirac function. After making the Fourier transformation to the variable $x \in R^n$ in (6), we obtain the following ordinary differential equation for $\widehat{E}$:

$$\begin{cases} (\widehat{E})_t + |\xi|^2(\widehat{E})_t + \eta|\xi|^2\widehat{E} + \sqrt{-1}(\beta\cdot\xi)\widehat{E} = 0, \\ \widehat{E}|_{t=0} = 1, \end{cases} \tag{7}$$

where ξ corresponds to $D_x = \frac{1}{\sqrt{-1}}(\partial x_1,\ldots,\partial x_n)$. By direct calculation, we have

$$\widehat{E}(\xi,t) = e^{\lambda(\xi)t}, \tag{8}$$

with

$$\lambda(\xi) \equiv \frac{-\eta|\xi|^2 - \sqrt{-1}(\beta\cdot\xi)}{1+|\xi|^2} \tag{9}$$

Some properties of the solutions $E(x,t)$ can be obtained as follows.

Lemma 2.2 [6]. *For any fixed* ε *there exists a positive constant C, such that*

$$\begin{aligned} &|\xi^\alpha\widehat{E}_L(\xi,t)| \leqslant C(1+t)^{-\frac{|\alpha|}{2}}, \\ &\int_{R^n}|\xi^\alpha\widehat{E}_L(\xi,t)|^2d\xi \leqslant C(1+t)^{-|\alpha|-\frac{n}{2}}. \end{aligned} \tag{10}$$

The proof of Lemma 2.2 can be seen in [6]. By using this lemma, we get the following proposition about the estimates on u_L.

The solution of (5) can be formulated by the Duhamel principle as

$$u_L(x,t) = (E_L * (\chi(D)u_0))(x,t) + \int_0^t E_L(\cdot,t-s) * \Phi(u(\cdot,s))ds. \tag{11}$$

Since $\|D_x^\alpha(E_L*(\chi(D)u_0))\|_{L^2}\leqslant\|D_x^\alpha E_L\|_{L^2}\|(\chi(D)u_0)\|_{L^1}$, by using Lemma 2.2, one has

$$\|D_x^\alpha(E_L*(\chi(D)u_0))\|_{L^2}\leqslant C(1+t)^{-\frac{|\alpha|}{2}-\frac{n}{4}}\|u_0\|_{L^1}. \tag{12}$$

For the second term of (11), first

$$\begin{aligned}\int_0^t\|D_x^\alpha E_L(\cdot,t-s)*\Phi(u(\cdot,s))\|_{L^2}ds&=\int_0^{t/2}\|D_x^\alpha E_L(\cdot,t-s)*\Phi(u(\cdot,s))\|_{L^2}ds\\&\quad+\int_{t/2}^t\|E_L(\cdot,t-s)*D_x^\alpha\Phi(u(\cdot,s))\|_{L^2}ds\\&\leqslant C\int_0^{t/2}\|D_x^\alpha E_L(\cdot,t-s)*\Phi(u(\cdot,s))\|_{L^1}ds\\&\quad+C\int_{t/2}^t\|E_L(\cdot,t-s)D_x^\alpha\Phi(u(\cdot,s))\|_{L^1}ds.\end{aligned}$$

Let

$$I(t)=\sup_{0\leqslant s\leqslant t,|\alpha|\leqslant m+1}\|D^\alpha u(\cdot,s)\|_{L^2}(1+s)^{\frac{|\alpha|}{2}+\frac{n}{4}}. \tag{13}$$

By (13) and Sobolev's embedded theorem, we have

$$\begin{aligned}&\sup_{|\alpha|\leqslant m+1}\|D^\alpha u(\cdot,t)\|_{L^2}\leqslant(1+t)^{-\left(\frac{|\alpha|}{2}+\frac{n}{4}\right)}I(t),\\&\sup_{|\alpha|\leqslant m-[\frac{n}{2}]}\|D^\alpha u(\cdot,t)\|_{L^\infty}\leqslant(1+t)^{-\left(\frac{|\alpha|}{2}+\frac{n}{4}\right)}I(t).\end{aligned} \tag{14}$$

Then, by using the Bernstein inequality, we have

$$\begin{aligned}&\|D_x^\alpha\Phi(u(\cdot,s))\|_{L^1}\leqslant C\varepsilon\|D_x^\alpha\varphi(u(\cdot,s))\|_{L^1}\\&\quad\leqslant C\varepsilon\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|}\|D_x^{\alpha_1}u(\cdot,s)\|_{L^2}\|D_x^{\alpha_2}u(\cdot,s)\|_{L^2}\|R_{\alpha_3}(u^{k-2})\|_{L^\infty},\end{aligned}$$

where

$$R_{\alpha_3}(u^{k-2})=\prod_{\sum_{j=1}^{k-2}\beta_j=\alpha_3}D_x^{\beta_j}u,$$

and $\beta_j\leqslant\min(\alpha_1,\alpha_2)$. By (14), we have

$$\begin{aligned}\|D_x^\alpha\Phi(u(\cdot,s))\|_{L^1}&\leqslant C\varepsilon\|D_x^\alpha\varphi(u(\cdot,s))\|_{L^1}\\&\leqslant C\varepsilon(1+s)^{-\frac{|\alpha|}{2}-\frac{n}{4}}I^k(t).\end{aligned} \tag{15}$$

Thus, by using Lemma 2.1 and (15), we get

$$\begin{aligned}&\int_0^t\|D_x^\alpha E_L(\cdot,t-s)*\Phi(u(\cdot,s))\|_{L^2}ds\\&\quad\leqslant C\varepsilon\left(\int_0^{t/2}(1+t-s)^{-\frac{|\alpha|}{2}-\frac{n}{4}}(1+s)^{-\frac{n}{2}}ds\right.\\&\quad\left.+\int_{t/2}^t(1+t-s)^{-\frac{n}{4}}(1+s)^{-\frac{|\alpha|}{2}-\frac{n}{2}}ds\right)I^k(t)\\&\quad\leqslant C\varepsilon(1+t)^{-\frac{|\alpha|}{2}-\frac{n}{4}}I^k(t).\end{aligned} \tag{16}$$

Summing up (11), (12), and (16), we get

$$\|D^\alpha u_L(t)\|_{L^2}\leqslant C(1+t)^{-\frac{|\alpha|}{2}-\frac{n}{4}}(\|u_0\|_{L^1}+\varepsilon I^k(t)). \tag{17}$$

3. The estimates for short wave

In this section, we will establish some L_2 estimates for the short wave by using the energy method. By applying the operator $1-\chi(D)$ to (1), we obtain

$$(u_S)_t-\Delta(u_S)_t-\eta\Delta u_S+(\beta\cdot\nabla)u_S+div((1-\chi(D))f(u))=0, \tag{18}$$

by integrating its product with u_S over $R^n\times[0,t]$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big((\|u_S(t)\|_{L^2}^2+\|\nabla u_S(t)\|_{L^2}^2)\Big)+\eta\|\nabla u_S(t)\|_{L^2}^2\\&\quad+\int_{R^n}((\beta\cdot\nabla)u_S+div((1-\chi(D))f(u)))u_S\,dx=0,\end{aligned} \tag{19}$$

since

$$\int_{R^n}((\beta\cdot\nabla)u_S)u_S\,dx=0,$$

and

$$\int_{R^n}(div(f(u)))u\,dx=0,$$

one has

$$\begin{aligned}&\int_{R^n}((\beta\cdot\nabla)u_S+div((1-\chi(D))f(u)))u_S\,dx\\&\quad=-\int_{R^n}(div(\chi(D))f(u))u_L\,dx\\&\qquad-\int_{R^n}((div((1-\chi(D))f(u))))u_L\,dx\\&\qquad-\int_{R^n}(div(\chi(D))f(u))u_S\,dx=R_1+R_2+R_3.\end{aligned}$$

Using the Bernstein inequality and (14), we get

$$\begin{aligned}|R_1|&\leqslant C\|div(\chi(D))f(u)\|_{L^2}\|u_L\|_{L^2}\\&\leqslant C\varepsilon^{n/2}\|div(\chi(D))f(u)\|_{L^1}\|u_L\|_{L^2}\\&\leqslant C\varepsilon^{n/2}\|div\,u\|_{L^2}\|u\|_{L^2}^2\|u^{k-2}\|_{L^\infty}\\&\leqslant C\varepsilon^{n/2}(1+t)^{-\frac{1}{2}-\frac{n(k+1)}{4}}I^{k+1}(t).\end{aligned}$$

For R_2, set

$$\chi_1(\xi)=\begin{cases}1, & |\xi|<2\varepsilon,\\0, & |\xi|>3\varepsilon,\end{cases}$$

as a smooth cut-off function. It is easy to see that $(1-\chi(\xi))\chi(\xi)\leqslant\chi_1(\xi)\chi(\xi)$.

Thus,

$$\begin{aligned}|R_2|&\leqslant C\|div(\chi_1(D))f(u)\|_{L^2}\|u_L\|_{L^2}\\&\leqslant C\varepsilon^{n/2}\|div(\chi_1(D))f(u)\|_{L^1}\|u_L\|_{L^2}\\&\leqslant C\varepsilon^{n/2}\|div\,u\|_{L^2}\|u\|_{L^2}^2\|u^{k-2}\|_{L^\infty}\\&\leqslant C\varepsilon^{n/2}(1+t)^{-\frac{1}{2}-\frac{n(k+1)}{4}}I^{k+1}(t).\end{aligned}$$

For R_3, similar to R_1, one has

$$|R_3|\leqslant C\varepsilon^{n/2}(1+t)^{-\frac{1}{2}-\frac{n(k+1)}{4}}I^{k+1}(t).$$

In summary, by the above inequalities, we obtain

$$\left|\int_{R^n}((\beta\cdot\nabla)u_S+div((1-\chi(D))f(u))u_S\,dx\right| \leqslant C\varepsilon^{n/2}(1+t)^{-\frac{1}{2}-\frac{n(k+1)}{4}}I^{k+1}(t). \quad (20)$$

By using Plancherel's theorem

$$\|\nabla u_S(t)\|_{L^2}^2=\|\xi(1-\chi(\xi))\widehat{u}(t)\|_{L^2}^2\geqslant\varepsilon^2\|u_S(t)\|_{L^2}^2. \quad (21)$$

Then, taking $\mu\leqslant\min\{1,\varepsilon^2\}\frac{\eta}{4}$, one has

$$-\mu(\|u_S(t)\|_{L^2}^2+\|\nabla u_S(t)\|_{L^2}^2)+\eta\|\nabla u_S(t)\|_{L^2}^2\geqslant 0. \quad (22)$$

Combining (19), (20), and (22) gives

$$\frac{1}{2}\frac{d}{dt}\left(e^{\mu t}(\|u_S(t)\|_{L^2}^2+\|\nabla u_S(t)\|_{L^2}^2)\right) \leqslant C\varepsilon^{n/2}e^{\mu t}(1+t)^{-\frac{1}{2}-\frac{n(k+1)}{4}}I^{k+1}(t).$$

Thus

$$\left(\|u_S(t)\|_{L^2}^2+\|\nabla u_S(t)\|_{L^2}^2\right)\leqslant Ce^{-\mu t}\left((\|u_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2)\right) +C\varepsilon^{n/2}\left(\int_0^t e^{-\mu(t-s)}(1+s)^{-\frac{1}{2}-\frac{n(k+1)}{4}}ds\right)I^{k+1}(t).$$

Setting $\theta(t)=e^{-\mu t}(1+t)^{1+\frac{n}{2}}$, we get

$$\left((1+t)^{\frac{n}{2}}\|u_S(t)\|_{L^2}^2+(1+t)^{1+\frac{n}{2}}\|\nabla u_S(t)\|_{L^2}^2\right) \leqslant C\theta(t)\left(\|u_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2\right)+\varepsilon^{n/2}I^{k+1}(t). \quad (23)$$

Here $\theta(0)=1$, and it is easy to see that there exists $t_1>0$ such that $\theta(t)<1$, when $t>t_1$. Thus, we can obtain from (17) and (23) that

$$\left((1+t)^{\frac{n}{2}}\|u(t)\|_{L^2}^2+(1+t)^{1+\frac{n}{2}}\|\nabla u(t)\|_{L^2}^2\right) \leqslant C\left(\|u_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^2+\|u_0\|_{L^1}^2\right)+\varepsilon^{n/2}I^{k+1}(t)+\varepsilon^2I^{2k}(t). \quad (24)$$

In order to obtain the estimates of $\|\nabla u_s(t)\|_{L^2}^2$, taking the inner product of (18) with $-\Delta u_S$ yields

$$(-(u_S)_t+\Delta(u_S)_t+\eta\Delta u_S-(\beta\cdot\nabla)u_S - div(1-\chi(D))(f(u)),\Delta u_S)=0. \quad (25)$$

It is easy to see that

$$(-(u_S)_t+\Delta(u_S)_t+\eta\Delta u_S,\Delta u_S) =\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_S\|^2+\|\Delta u_S\|^2\right)+\eta\|\Delta u_S\|^2.$$

Since

$$(-(\beta\cdot\nabla)u_S,\Delta u_S)=0,$$

and

$$|(div(1-\chi(D))(f(u)),\Delta u_S)|\leqslant\frac{\eta}{2}\|\Delta u_S\|_{L^2}^2+C_\eta\|div(1-\chi(D)) (f(u))\|_{L^2}^2,$$

we have

$$\frac{d}{dt}\left(\|\nabla u_S\|_{L^2}^2+\|\Delta u_S\|_{L^2}^2\right)+\eta\|\Delta u_S\|_{L^2}^2 \leqslant 2C_\eta\|div(1-\chi(D))(f(u))\|_{L^2}^2.$$

As (21), by using Plancherel's theorem

$$\|\Delta u_S\|_{L^2}^2\geqslant\varepsilon^2\|\nabla u_S\|_{L^2}^2.$$

Taking $\mu\leqslant\min\{1,\varepsilon^2\}\frac{\eta}{4}$, we have

$$\|\nabla u_S(t)\|_{L^2}^2+\|\Delta u_S(t)\|_{L^2}^2 \leqslant Ce^{-\mu t}\left((\|\nabla u_S(0)\|_{L^2}^2+\|\Delta u_S(0)\|_{L^2}^2)\right) +2C_\eta\int_0^t e^{-\mu(t-s)}\|div(1-\chi(D))(f(u(s)))\|_{L^2}^2ds. \quad (26)$$

Since

$$\|div(1-\chi(D))(f(u(s)))\|_{L^2}^2\leqslant C\|div\,u\|_{L^2}^2\|u^{k-1}\|_{L^\infty}^2, \quad (27)$$

by using (24), (26), and (27), we obtain

$$\left((1+t)^{1+\frac{n}{2}}\|\nabla u_S(t)\|_{L^2}^2+(1+t)^{2+\frac{n}{2}}\|\Delta u_S(t)\|_{L^2}^2\right) \leqslant C\left((\|u_0\|_{H^2}^2+\|u_0\|_{L^1}^2)+\varepsilon^{n/2}I^{k+1}(t)+\varepsilon^2I^{2k}(t)\right). \quad (28)$$

Thus, (28) and (17) also give

$$\left((1+t)^{1+\frac{n}{2}}\|\nabla u(t)\|_{L^2}^2+(1+t)^{2+\frac{n}{2}}\|\Delta u(t)\|_{L^2}^2\right) \leqslant C\left((\|u_0\|_{H^2}^2+\|u_0\|_{L^1}^2)+\varepsilon^{n/2}I^{k+1}(t)+\varepsilon^2I^{2k}(t)\right). \quad (29)$$

For a similar method, for any $0\leqslant h\leqslant m$, we have

$$\left((1+t)^{\frac{n}{2}+h}\|\partial^h u(t)\|_{L^2}^2+(1+t)^{\frac{n}{2}+h+1}\|\partial^{h+1}u_S(t)\|_{L^2}^2\right) \leqslant C\theta(t)\left((\|u_0\|_{H^{h+1}}^2+\|u_0\|_{L^1}^2)+\varepsilon^{n/2}I^{k+1}(t)+\varepsilon^2I^{2k}(t)\right). \quad (30)$$

Here, $\partial^l=\sum_{|\alpha|=l}\partial^\alpha$.

Now we come back to prove Theorem 1.1 by using (30). In fact

$$I^2(t)\leqslant\sum_{|\alpha|\leqslant m+1}(1+t)^{\frac{n}{2}+|\alpha|}\|\partial^\alpha u\|_{L^2}^2. \quad (31)$$

By combining (30) and (31), we have

$$I^2(t)\leqslant C\left(\|u_0\|_{H^{m+1}}^2+\|u_0\|_{L^1}^2+\varepsilon^2(I^{k+1}(t)+I^{2k}(t))\right). \quad (32)$$

Let $M^2=4C(\|u_0\|_{H^{m+1}}^2+\|u_0\|_{L^1}^2)>0$. Since C is a constant independent of ε and t, and the above inequality holds for any $\varepsilon>0$, taking $\varepsilon^2=(2C(M^{2k-2}+M^{k-1}))^{-1}$, we see that

$$I^2(0)\leqslant 2C\left(\|u_0\|_{H^{m+1}}^2+\|u_0\|_{L^1}^2\right)=\frac{1}{2}M^2. \quad (33)$$

We will prove that $I^2(t)\leqslant M^2$ for all $t>0$. For this, we define

$$T=\sup\{t\geqslant 0: I^2(t)\leqslant M^2\}.$$

If $T<\infty$, then by the definition we find $I^2(t)=M^2$, then

$$M^2 \leqslant \frac{1}{4}M^2 + \varepsilon^2(M^{2k} + M^{k+1}) \leqslant \frac{3}{4}M^2. \tag{34}$$

It then follows that $M^2 < 0$; the contradiction tells us that $T = \infty$ and therefore

$$I^2(t) \leqslant 4C\left(\|u_0\|^2_{H^{m+1}} + \|u_0\|^2_{L^1}\right). \tag{35}$$

From (35) and the definition of $I(t)$, we know that Theorem 1.1 is valid.

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References

[1] Albert J. On the decay of solutions of the generalized Benjamin–Bona–Mahony equation. J Math Anal Appl 1989;141:527–37.
[2] Avrin J, Goldstein JA. Global existence for the Benjamin–Bona–Mahony equations. Nonlinear Anal 1985;9:861–5.
[3] Benjamin TB, Bona JL, Mahony J. Model equations for long waves in nonlinear dispersive systems. Philos Trans Roy Soc London 1972;272:47–78.
[4] Biler P. Long time behavior of solutions of the generalized Benjamin–Bona–Mahony equation in two-space dimensions. Differential Integral Equations 1992;5:891–901.
[5] Fang SM, Guo BL. Long time behavior for solution of initial-boundary value problem for one class of system with multidimensional inhomogeneous GBBM equations. Appl Math Mech 2005;26(6):665–75.
[6] Fang SM, Guo BL. The decay rates of solutions of generalized Benjamin–Bona–Mahony equations in multi-dimensions. Nonlinear Anal 2008;69(7):2230–5.
[7] Goldstein JA, Wichnoski BJ. On the Benjamin–Bona–Mahony equation in higher dimensions. Nonlinear Anal 1980;4:861–5.
[8] Guo BL. Initial boundary value problem for one class of system of multidimensional inhomogeneous GBBM equations. Chin Ann Math Ser B 1987;8(2):226–38.
[9] Zhang LH. Decay of solutions of generalized Benjamin–Bona–Mahony equation in n-space dimensions. Nonlinear Anal 1995;25:1343–69.